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A STUDY OF THE INTERACTION OF A SUPERSONIC
JET AND AN ACOUSTIC FIELD

M. G. Lebedev and G. F. Telenin
(Moscow)

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A STUDY OF THE INTERACTION OF A SUPERSONIC
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M. G. Lebedev and G. F. Telenin

ABSTRACT. The theory of Pulsations in the wake of a supersonic nozzle is developed. The mechanism of this phenomenon is discussed on the basis of previous experimental and theoretical works.

The phenomenon of pulsation of a supersonic jet emitted from a nozzle in /82* an immersed space is studied. Based on an analysis of experimental data, a scheme is proposed for this phenomenon and a mathematical description is given of the problem corresponding to the proposed scheme. A survey of fundamental findings on the problem can be found in [1]. Reference [2] is taken as a basis for analysis of the phenomenon, as it gives systematic data. A general picture of the phenomenon consists of the following. For a given nozzle M number and pressure in the surrounding medium p_e , there is a pressure variation range in the receiver p_0 at which the jet begins to pulse with a given frequency $f = \omega/2\pi$ and drops at a certain distance from the nozzle section (considerably earlier than in the absence of oscillations). At the same time, in the surrounding medium acoustic waves are propagated to the nozzle section at the same frequency. The equivalence of the frequencies means that the wave source is stationary in space, since there is no Doppler effect. Basically, the radiation of acoustic waves takes place from the jet breakdown area. These waves are nearly spherical in form and the oscillations on various sides of

*Numbers in the margin indicate pagination in the original foreign text.

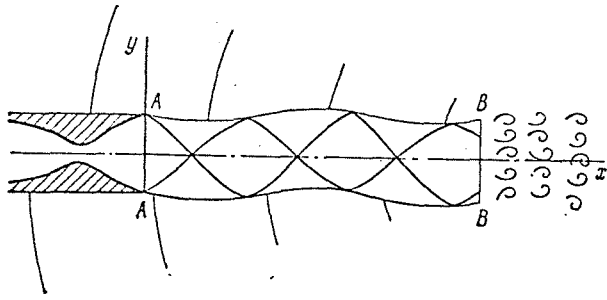


Figure 1.

the jet are in opposite phase. Schematically, this picture is shown in Figure 1.

Supersonic jet pulsations with a "direct" frequency were discovered by Powell [3] as early as 1953; he proposed the first scheme explaining the phenomenon. Although the basic concept

of an inverse relationship through an external acoustic field is clearly correct, Powell's model has several shortcomings, and is not the only possible one.

1. Scheme of the Phenomenon

When a fluid or gas flow is not homogeneous, under certain conditions it can lose its stability. A mechanism is created which converts a part of the energy of the forward motion into pulsations. As a rule, the oscillations which occur have a broad frequency spectrum and a complex, random character. However, with poorly streamlined bodies or through other causes, causing a complex flow structure, as a result of the nonlinear development of disturbances after the loss of stability there sometimes occur rather regular fluctuations with a definite frequency.

The simplest scheme for such phenomena is given by the flow around an empty body acting as a resonator. For the sake of simplicity, let us consider a Helmholtz resonator (Figure 2). Because of the loss of stability of section ABC at the mouth of the resonator, a pressure pulsation occurs. Let ξ be the displacement of a gas mass m , contained in a narrow channel BD. The oscillation equation will then take the form

$$\xi'' + 2\eta\xi' + \omega_0^2\xi = m^{-1}S\Delta p \quad (\omega_0 = c(S/LV)^{1/2}) \quad (1.1)$$

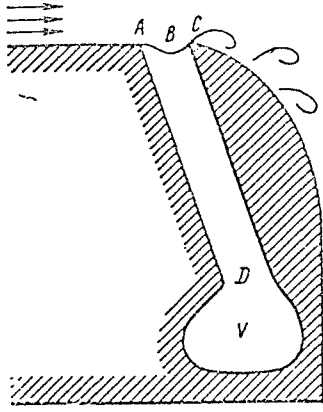


Figure 2.

Here η is the damping coefficient of the resonator, ω_0 , S and L are the cross sectional area and effective length of the channel, V is the volume of the cavity of the resonator and c /83 is the speed of sound inside it. Δp is the variable pressure distance from outside the mouth of the channel. Let the pressure disturbance be represented by the harmonic oscillation

$$\Delta p = P \cos \omega t, \quad (1.2)$$

Assuming that the gas compression in the resonator occurs adiabatically, for the pressure disturbance in the cavity, we have

$$\delta p = P \omega_0^2 |H(i\omega)| \cos \omega t, \quad H(i\omega) = (\omega_0^2 - \omega^2 + i2\eta\omega)^{-1} \quad (1.3)$$

Here $H(i\omega)$ is the amplitude-frequency characteristic of the oscillatory system of (1.1), and $\omega_0 |H(i\omega)|$ is the coefficient of disturbance amplification. Actually, the pressure disturbance Δp at the mouth of the resonator is not represented by a harmonic oscillation with a given frequency, but by a random process with a broad spectrum, characterized by the spectral density $w_{\Delta p}(\omega)$. The spectral density of pulsations in the cavity, in accordance with (1.3), is given by the formula

$$w_{\delta p}(\omega) = \omega_0^4 |H(i\omega)|^2 w_{\Delta p}(\omega) \quad (1.4)$$

Thus, if the oscillatory system has a high Q factor ($\omega_0/2\eta \gg 1$), then the resonator is represented by a narrow-band filter which selects the oscillatory energy of the disturbances at short frequency intervals of approximately ω_0 .

If there is an inverse relationship between the oscillations in the resonator and the intensity of pressure pulsations in the mouth of the resonator in the frequency interval of approximately ω_0 and if there is a mechanism limiting the pulsation intensity increase, the "resonator-external flow" system will be auto-oscillatory.

In a certain sense, the oscillating supersonic jet in the section from the opening of the nozzle to the breakdown area can be compared to a resonator. In Figure 1, let us consider the jet section AB. The waves, moving along the surface of the jet

$$\Delta p = \cos(\omega t + 2\pi x / l) = \cos(\omega t + \omega x / a_e) \quad (1.5)$$

(l , ω , a_e are the wave length, angular frequency and speed of sound in the external medium) transfer the oscillatory energy to the jet. Energy diffusion is accomplished by the transfer of oscillatory energy across section BB and the radiation to the external medium of the acoustic waves of oscillating jets from surface AB. By studying the quasi-periodicity along x of the geometrical structure of the jet and the nature of disturbance distribution in the supersonic flow, it is natural to expect that in general disturbances in the jet which are caused by a periodic wave on surface AB will be quasi-periodic along x . Then, taking a wave length $l = l_*$ or a frequency $\omega = \omega_*$, we can synchronize the wave on the surface with the propagation of disturbances in the jet, so that both positive and negative pressure disturbances in the jet build up along the jet as far as the breakdown region, exceeding the disturbances in the wave on the surface, so that disturbances in the jet are amplified. The jet will serve as a filter (or resonator) removing oscillatory energy from the external acoustic field in a short frequency interval of approximately ω_* .

Under real conditions, the jet leaving the nozzle can be unrated (under- or over-expanded). The excess or insufficient pressure drop operates basically in the initial section of the jet, but further on its average cross sectional

size is determined by the total pressure drop $p_0 - p_e$ between the receiver and external medium. Let us now introduce the average diameter D and M number /84 of the jet which determine its size and geometric structure behind the initial section. Let D_m be the diameter of the critical section of the nozzle; according to the theory of homogeneous gas flows M and $D/D_m = F(M)$ are expressed by means of the pressure ratio p_e/p_0 . Consequently, according to the scheme for the phenomenon considered we have, based on the condition of synchronization of a wave moving over the surface,

$$\frac{l_*}{D_m} = F_1\left(M, \frac{a}{a_e}\right), \quad S_* = \frac{\omega_* D}{2\pi a_e} = \frac{D}{D_m} \frac{D_m}{l_*} = F_2\left(M, \frac{a}{a_e}\right) \quad (1.6)$$

where a is the average speed of sound in the jet, S_* is the critical Strouhal number. If the gas temperatures in the receiver and the external medium are fixed, the ratio a/a_e for variable p_0 depends only on M . Experiments with cold jets carried out for different values of M_A — the geometric Mach number, computed from the area ratios at the nozzle output and at the critical section — show that in accordance with (1.6) S_* is basically determined by the parameter M (see Figure 12 in [2]). Since for a fixed p_0/p_e and a variable M_A the unrated characteristics of the jet vary, from these results it also follows that S_* depends slightly on the discontinuity intensity. Experiments have shown the dependence of S_* on the degree of heating of the jet, and consequently, on the ratio a/a_e , which also agrees with (1.6).

For general reasons and as a result of experimental data (the frequency of the external acoustic waves is equal to the pulsation frequency in the jet) it is natural that the intensity of waves emitted into the external medium from the breakdown region of the jet is connected to the pulsation intensity in the jet, i.e., the oscillatory energy radiates basically in the frequency interval of approximately ω . One of the conceivable mechanisms, which limits the growth of pulsations, consists of the following. As the intensity of external waves and oscillations of the jet increase, the greater is the disturbance of the jet geometric structure (which is quasi-periodic along x) and,

as a result, of its filtering properties. Thus, the growth of disturbances toward the breakdown region is stopped. Within the framework of the scheme for the phenomenon considered here, it is natural to assume that the discontinuity intensity and the processes connected with viscosity and thermal conductivity must influence the nonlinear mechanism of acoustic energy radiation from the breakdown region of the jet (possibly from the area of discontinuity intersection with the free surface also). This agrees with experimental data showing the effects of the unrated characteristics of the jet, the degree of heating and, probably, the Reynolds number on the intensity of narrow-band pulsations in the external field [1]. Let us note that experimental data on the effects of subsonic wake are consistent with the scheme given.

In summary, we can say that the scheme considered for the occurrence and continuance of narrow-band pulsations during the outflow of a single supersonic jet agrees in general with known experimental data. Naturally, this is not proof of its correctness. The question of clarifying the mechanism of narrow-band pulsations will require further theoretical and experimental research.

2. Statement of the Mathematical Problem

In this work, we shall discuss the questions dealing with the nonlinear mechanism of acoustic energy radiation from the breakdown region of the jet. We shall turn our attention to a study of the amplitude-frequency characteristics of the jet, which determines the basic frequency of the narrow-band pulsations. We shall therefore consider interaction of a supersonic jet with a given external acoustic field. To simplify the mathematical problem, we shall consider an idealized scheme of the phenomenon. The external acoustic waves will be considered plane. In accordance with experimental results, the oscillations in them are static. In the case of an axi-symmetrical jet, the field of pressure disturbances in a cylindrical coordinate system x, y, ψ can be /85
stated in the form

$$p' = A(t) \cos \left[\omega t + \frac{2\pi}{l} x + B(t) \right] \cos[\psi + C(t)] \quad (2.1)$$

where A, B, C are random functions whose laws require further study. In order to determine the interaction of a random acoustic field (2.1) with the jet, it is sufficient to look at its interaction with the determined field

$$p' = P \cos \left(\omega t + \frac{2\pi}{l} x \right) \cos \psi \quad (2.2)$$

where P is a constant. The solution of this problem is analogous to the determination of the amplitude-frequency characteristics of a resonator in an autooscillatory system with an open feedback (see above). Since the intensity of the discontinuities does not affect the basic frequency of the narrow-band pulsations, we shall consider the rated characteristics of a jet, and disturbances in the external acoustic field and jet will be assumed to be small. This allows us to consider the flow as potential and to solve the problem in the linear formulation. If we introduce the potentials Φ and Φ_e of disturbances with velocities V' and V_e' separately for the jet and the external medium according to formulas

$$V' = a \text{ grad } \Phi, \quad V_e' = a_e \text{ grad } \Phi_e$$

then they will be satisfied by the equations

$$\begin{aligned} \beta^2 \frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} - \frac{1}{y^2} \frac{\partial^2 \Phi}{\partial \psi^2} - \frac{1}{y} \frac{\partial \Phi}{\partial y} + \frac{2M}{a} \frac{\partial^2 \Phi}{\partial x \partial t} + \frac{1}{a^2} \frac{\partial^2 \Phi}{\partial t^2} &= 0 \\ \frac{\partial^2 \Phi_e}{\partial x^2} + \frac{\partial^2 \Phi_e}{\partial y^2} + \frac{1}{y^2} \frac{\partial^2 \Phi_e}{\partial \psi^2} + \frac{1}{y} \frac{\partial \Phi_e}{\partial y} - \frac{1}{a_e^2} \frac{\partial^2 \Phi_e}{\partial t^2} &= 0 \quad (\beta^2 = M^2 - 1) \end{aligned} \quad (2.3)$$

Then the pressure disturbances in the jet and in the external medium are defined by the relationships

$$\begin{aligned} p' &= -\gamma p \frac{R}{a} \frac{d\Phi}{dt} = -\gamma p \frac{R}{a} \left(\frac{\partial \Phi}{\partial t} + aM \frac{\partial \Phi}{\partial x} \right) \\ p_e' &= -\gamma_e p \frac{R}{a_e} \frac{d\Phi_e}{dt} = -\gamma_e p \frac{R}{a_e} \frac{\partial \Phi_e}{\partial t} \end{aligned}$$

where p is the undisturbed pressure which is the same in the jet and in the external medium, γ and γ_e are adiabatic exponents, R is a characteristic linear dimension. The simplest statement of the problem is given in Figure 3. From a semi-infinite rigid cylinder ($-\infty \leq x < 0$) with unit radius there flows a semi-infinite cylindrical supersonic jet ($0 < x \leq \infty$). From $x = \infty$ along the direction of the cylinder plane waves are radiated (2.2). A solution is /86 sought in the $-\infty \leq x \leq \infty$, $0 \leq y \leq \infty$ region. The boundary conditions are the conditions where there is no flow to the external and internal surfaces of the cylinder

$$\frac{\partial \Phi_e}{\partial y} = \frac{\partial \Phi}{\partial y} = 0 \quad \text{for } -\infty \leq x < 0, \quad y = 1 \quad (2.4)$$

and the conditions of equal pressure and displacement of the interface between the jet and the external medium

$$\frac{\gamma}{a} \left(\frac{\partial \Phi}{\partial t} + aM \frac{\partial \Phi}{\partial x} \right) = \frac{\gamma_e}{a_e} \frac{\partial \Phi_e}{\partial t}, \quad \frac{1}{a_e} \frac{\partial^2 \Phi}{\partial t \partial y} = \left(\frac{1}{a} \frac{\partial}{\partial t} + M \frac{\partial}{\partial x} \right) \frac{\partial \Phi_e}{\partial y} \quad (2.5)$$

for $0 < x \leq \infty$, $y = 1$ and, finally, the radiation condition which requires that for $y \ll 1$ the solution be expressed as departing waves.

The boundary problem formulated can be solved, for example, by the Wiener-Hopf method. However, as a result of solving such a problem, the displacement of the jet surface not only does not decrease, but may even increase. Therefore, the interaction of the jet with the external acoustic field in the section $x > L$ causes finite disturbances in the $x < L$ region. In fact, effective interaction between jets and the external acoustic wave occurs only in the portion from the nozzle opening to the breakdown region of the jet (section AB in Figure 1 and $0 \leq x \leq L$ in Figure 3). Thus, a solution in the $x < L$ region is of greatest interest, but gives a distorted result. To eliminate this shortcoming, the formulation of the problem must be altered so that for $x > L$ the jet surface displacement decreases rapidly to zero. Then the boundary

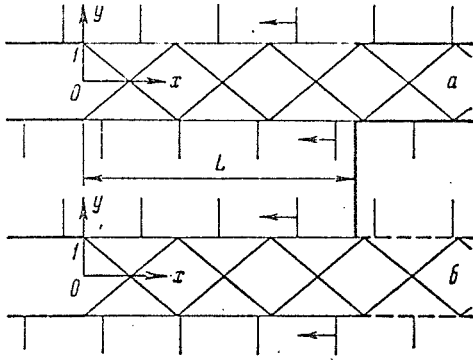


Figure 3.

conditions (2.5) function in the interval $0 < x \leq L$, while for the remaining portions, the boundary conditions take the form

$$\begin{aligned} \partial\Phi_e / \partial y = \varphi(x) & \quad \text{for} \\ L < x \leq \infty, & \quad y = 1. \end{aligned} \quad (2.6)$$

where $\phi(x)$ decreases rather quickly to zero as the function increases

(dotted portion in Figure 3 b). The

other boundary conditions remain in force. The existence of three regions for x with different boundary conditions leads to a problem which cannot be solved analytically, but must be solved numerically.

If we neglect the radiation of acoustic energy from the jet surface into surrounding space, and consequently, the interaction of the jet with the external medium, we obtain a simplified statement of the problem while retaining, however, its basic properties. In this statement, from the semi-infinite cylinder ($-\infty \leq x < 0$) there is emitted a semi-infinite supersonic jet ($0 < x \leq \infty$), on whose surface the traveling pressure wave is given

$$\Delta p = P \cos \left(\omega t + \frac{2\pi}{l} x \right) \cos \psi \quad (2.7)$$

Since there is no interaction with the external medium, and consequently, there is no transfer of disturbances above the flow, we may seek a solution for the whole infinite jet. The boundary conditions take the form

$$\begin{aligned} \partial\Phi / \partial y = 0 & \quad \text{for} \quad -\infty \leq x < 0, \quad y = 1 \\ p' = \Delta p & \quad \text{for} \quad 0 \leq x \leq \infty, \quad y = 1 \end{aligned} \quad (2.8)$$

Simple analysis shows that the solutions for the problem formulated in the plane and axisymmetric cases have identical properties. Therefore, we

shall solve the plane problem. To obtain more general results, let us assume pressure disturbances $p' = \Delta p$ at the upper and lower limits of the jet are asymmetric as well as symmetric with respect to the medium line of the jet.

3. Formulation of a Linear Solution

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The equation for the potential disturbance in the plane case can be written in the form

$$\beta^2 \frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} + 2M \frac{\partial^2 \Phi}{\partial \tau \partial x} + \frac{\partial^2 \Phi}{\partial \tau^2} = 0 \quad (3.1)$$

Here the nondimensional time $\tau = aR^{-1}t$ is introduced. We shall also use a dimensionless frequency $\nu = a^{-1}R\omega$. As the characteristic linear dimension R we shall take half the width of the jet, and we shall assume that $y = 0$ at its lower boundary (Figure 4). The conditions at the boundary of the jet have the form

$$\begin{aligned} p'_{y=0} &= -\gamma \left(\frac{\partial \Phi}{\partial \tau} + M \frac{\partial \Phi}{\partial x} \right)_{y=0} = P \cos(\nu\tau + \nu_e x) \\ p'_{y=2} &= -\gamma \left(\frac{\partial \Phi}{\partial \tau} + M \frac{\partial \Phi}{\partial x} \right)_{y=2} = jP \cos(\nu\tau + \nu_e x) \end{aligned} \quad (3.2)$$

where $\nu_e = \nu a/a_e$ and j is equal to 1 or -1 in the symmetrical or asymmetrical cases, respectively.

A solution for Equation (3.1) is sought in the form

$$\Phi = F_1(x, y) \cos(\nu\tau - \sigma x) + F_2(x, y) \sin(\nu\tau - \sigma x) \quad (3.3)$$

Substituting (3.3) in (3.1) and selecting σ so that the coefficients for $\partial F_1/\partial x$ and $\partial F_2/\partial x$ strive to zero, we find that the functions F_1 , F_2 satisfy one and the same equation

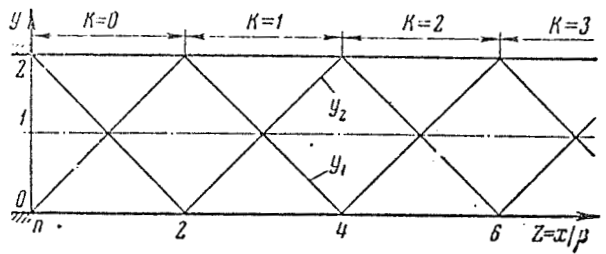


Figure 4.

$$\beta^2 \frac{\partial^2 F_k}{\partial x^2} - \frac{\partial^2 F_k}{\partial y^2} + \lambda^2 F_k = 0 \quad (3.4)$$

$$\left(\sigma = \frac{Mv}{\beta^2}, \lambda = \frac{v}{\beta}, k = 1, 2 \right)$$

The boundary conditions for $y = 0$ are written in the form

$$\begin{aligned} -\gamma \left[M \frac{\partial F_1}{\partial x} + (v - \sigma M) F_2 \right] &= P \cos(\sigma x + v_e x), \\ \gamma \left[M \frac{\partial F_2}{\partial x} - (v - \sigma M) F_1 \right] &= P \sin(\sigma x + v_e x) \end{aligned} \quad (3.5)$$

For $y = 2$ the right-hand terms (3.5) should be multiplied by j .

To solve system (3.4) we apply to each of the equations a Laplace transform [4] with respect to variable x

$$G_k(p, y) = \int_0^\infty F_k(x, y) e^{-px} dx \quad (3.6)$$

where p is a complex quantity. The function G_k is satisfied by the equations /88

$$\frac{d^2 G_k}{dy^2} - \mu^2 G_k = 0 \quad (\mu^2 = p^2 \beta^2 + \lambda^2) \quad (3.7)$$

The solution for (3.7) is

$$G_k = A_k(p) e^{\mu y} + B_k(p) e^{-\mu y} \quad (3.8)$$

The boundary conditions for System (3.7) from which we can find the constants A_k , B_k , are obtained by applying a Laplace transform to (3.5). Completely defining the function $G_k(p, y)$, we can find the inverse function $F_k(x, y)$ by using the inverse Laplace transform

$$F_h(x, y) = \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} G_h(p, y) e^{px} dp \quad (3.9)$$

where s is an arbitrary positive number. In its expanded form the solution for (3.9) is written

$$F_h = \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} G_{h0} \left[\frac{je^{2\mu} - 1}{e^{4\mu} - 1} e^{\mu y} + \frac{e^{2\mu} - j}{e^{4\mu} - 1} e^{-\mu y} \right] e^{\zeta x / \beta} d\zeta \quad (3.10)$$

where the functions G_{k0} are related to the boundary conditions and have the form

$$G_{10} = \frac{P\beta}{\gamma M} \frac{1}{d^2 + \zeta^2} \frac{cd - \zeta^2}{c^2 + \zeta^2}, \quad G_{20} = \frac{P\beta}{\gamma M} \frac{1}{d^2 + \zeta^2} \frac{\zeta(c + d)}{c^2 + \zeta^2} \quad (3.11)$$

$$\zeta = p\beta, \quad c = \beta(\sigma + v_e), \quad d = \lambda / M = v / \beta M$$

The integrands in (3.10) have no singularities in the complex half-plane $\text{Re } \zeta < s$ except for the simple poles at the points

$$\zeta = \pm id, \quad \zeta = \pm ic, \quad \zeta = \pm i[\lambda^2 + \pi^2(n + (j+1)/4)^2]^{1/2} \quad (n = 0, 1, \dots)$$

By applying Jordan's lemma and the Cauchy theorem of residues [4], we can calculate the integrals (3.10) and obtain a solution for the system (3.4).

$$F_1 = \frac{P\beta}{\gamma M} \left\{ \frac{1}{c-d} \left[\frac{e^{Dy} + je^{D(2-y)}}{1 + je^{2D}} \sin \frac{dx}{\beta} - \frac{(1+j)\cos C(1-y) + (1-j)\sin C(1-y)}{(1+j)\cos C + (1-j)\sin C} \sin \frac{cx}{\beta} \right] + 2 \sum_{n=0}^{\infty} \frac{(cd + \Lambda^2) \pi N \sin \pi Ny}{(c^2 - \Lambda^2)(d^2 - \Lambda^2)\Lambda} \sin \Lambda \frac{x}{\beta} \right\} \quad (3.12)$$

$$F_2 = \frac{P\beta}{\gamma M} \left\{ \frac{1}{c-d} \left[\frac{e^{D\nu} + je^{D(2-\nu)}}{1+je^{2D}} \cos \frac{dx}{\beta} - \frac{(1+j)\cos C(1-y) + (1-j)\sin C(1-y)}{(1+j)\cos C + (1-j)\sin C} \cos \frac{cx}{\beta} \right] + \right. \\ \left. + 2 \sum_{n=0}^{\infty} \frac{(c+d)\pi N \sin \pi N y}{(c^2 - \Lambda^2)(d^2 - \Lambda^2)} \cos \Lambda \frac{x}{\beta} \right\} \quad (3.12) \\ (\text{con't.})$$

$$C = (c^2 - \lambda^2)^{1/2}, \quad D = (\lambda^2 - d^2)^{1/2}, \quad N = n + (j+1)/4, \quad \Lambda = (\lambda^2 + \pi^2 N^2)^{1/2}$$

For known F_1, F_2 it is possible to calculate the disturbance of potential Φ , the longitudinal and transverse velocity components u' and v' and pressure p'

$$\Phi_1 = F_1 \cos \sigma x - F_2 \sin \sigma x, \quad \Phi_2 = F_1 \sin \sigma x + F_2 \cos \sigma x \\ u_1 = \partial \Phi_1 / \partial x, \quad u_2 = \partial \Phi_2 / \partial x, \quad v_1 = \partial \Phi_1 / \partial y, \quad v_2 = \partial \Phi_2 / \partial y \\ p_1 = -\gamma(M \partial \Phi_1 / \partial x + v \Phi_2), \quad p_2 = -\gamma(M \partial \Phi_2 / \partial x - v \Phi_1) \quad (3.13) \\ \Phi = \Phi_1 \cos v\tau + \Phi_2 \sin v\tau \dots$$

To calculate F_1, F_2 from Formulas (3.12) and the quantities defined by (3.13) a computer program was set up. There were 175 terms in the series included in the expressions for F_1, F_2 , which ensured high accuracy. We should note that these series converge nonuniformly; and thus the calculation of derivatives of functions F_1, F_2 by termwise differentiation of the series leads to difficulties. In this connection, the derivatives were calculated numerically in accordance with a second order scheme. The program was also used to calculate the amplitude of potential disturbances, velocities and pressure

$$K_\Phi = P^{-1}(\Phi_1^2 + \Phi_2^2)^{1/2}, \quad K_p = P^{-1}(p_1^2 + p_2^2)^{1/2}, \dots$$

The quantity K_p is the coefficient of pressure disturbance amplification at a certain point inside the jet.

In order to use the theorem and methods of operational calculation, by means of which explicit expressions for F_1, F_2 were obtained, the latter should satisfy a series of conditions. Without verifying that these conditions are

satisfied, solution (3.12) is purely formal. More simply, however, instead of making such a verification, one can substitute expressions for F_1, F_2 (3.12) in Equation (3.4) and the boundary conditions (3.5). One can immediately check whether or not these functions are the solutions sought. For this purpose, an operator was included in the computer program to check for fulfillment of (3.4) and (3.5); calculations showed that (3.4) and (3.5) were satisfied with high accuracy.

Let us define the critical frequencies v_* of Solution (3.12) at which we observe maximum increase in disturbances inside the jet in comparison with given disturbances at the boundary. It is natural to assume that the frequencies, at which Solution (3.12) has singularities, will be critical. Turning once more to the solution in a complex plane (3.10) and (3.11), we note that for specific conditions the poles of the integrands

$$\zeta = ic, \quad \zeta = i(\lambda^2 + \pi^2 N^2)^{1/2}$$

coincide to form a second-order pole. From this condition, we find an expression for the frequencies v_* in the asymmetric case ($j = -1$)

$$v_* = \pi n \left[\left(\frac{M}{\beta} + \frac{a}{a_e} \beta \right)^2 - \frac{1}{\beta^2} \right]^{-1/2} \quad (3.14)$$

and in the symmetric case ($j = 1$)

$$v_* = \pi \left(n - \frac{1}{2} \right) \left[\left(\frac{M}{\beta} + \frac{a}{a_e} \beta \right)^2 - \frac{1}{\beta^2} \right]^{-1/2} \quad (3.15)$$

where $n = 1, 2, \dots$

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An analysis of Solution (3.12) shows (see paragraph 4) that the frequencies, determined by Relations (3.14) or (3.15), are in fact critical. Figure 5 shows the dependence of the critical Strouhal number, calculated from the

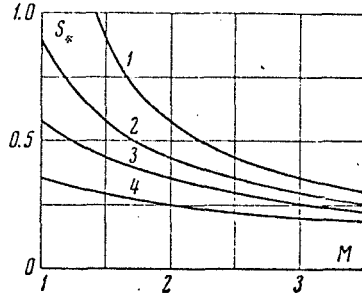


Figure 5.

speed of sound in the external medium a_e , the width of the jet and frequency $\omega/2\pi$ ($S_* = v_* a / \pi a_e$), on the M number of the undisturbed flow in the jet (curves 1, 2, 3, 4 correspond to the values $a/a_e = \infty, 2, 1, 0.5$) for the anti-symmetric case for $n = 1$.

Solution (3.12) may be simplified in the limiting case, when the ratio of the speeds of sound in the jet and in the external medium $a/a_e = \infty$. Physically, this corresponds to a strong heated jet. We shall introduce the notation $a = v/v_*$ ($n = 1$) and pass to the limit in (3.12). The series in (3.12) are then changed into Fourier series which are easily summed. In the end, we obtain (for the anti-symmetric case)

$$\begin{aligned} F_1 &= P\beta(\gamma M)^{-1}(\alpha\pi \sin \alpha\pi)^{-1}(-\sin \alpha\pi z \sin \alpha\pi(1-y) + \varphi_1) \\ F_2 &= P\beta(\gamma M)^{-1}(\alpha\pi \sin \alpha\pi)^{-1}(-\cos \alpha\pi z \sin \alpha\pi(1-y) + \varphi_2) \end{aligned} \quad (3.16)$$

$$\text{при } 0 \leq z - 2k \leq y$$

$$\varphi_1 = \sin \alpha\pi(z - 2k) \sin \alpha\pi(1 - y), \quad \varphi_2 = \cos \alpha\pi(z - 2k) \sin \alpha\pi(1 - y)$$

$$\text{при } y \leq z - 2k \leq 2 - y$$

$$\varphi_1 = \sin \alpha\pi(1 - z + 2k) \sin \alpha\pi y, \quad \varphi_2 = -\cos \alpha\pi(1 - z + 2k) \sin \alpha\pi y + \sin \alpha\pi$$

$$\text{при } 2 - y \leq z - 2k \leq 2$$

$$\varphi_1 = -\sin \alpha\pi(2 - z + 2k) \sin \alpha\pi(1 - y),$$

$$\varphi_2 = \cos \alpha\pi(2 - z + 2k) \sin \alpha\pi(1 - y)$$

Here $z = x/\beta$ and the parameter k , equal to the whole part of $z/2$, represents the "nucleus" number of the jet (Figure 4).

Differentiating (3.16) with respect to x and y and using (3.13), we find the dependence on the pressure coordinates and velocity components.

Let us here write the expression for the coefficient of pressure disturbance amplification K_p

$$K_p = \begin{cases} |\vartheta(k)| & (0 \leq z - 2k < y) \\ [1 - \vartheta(k)\vartheta(k+1) \sin \alpha\pi y (\sin \alpha\pi(1-y))^{-1}]^{1/2} & (y < z - 2k < 2 - y) \\ |\vartheta(k+1)| & (2 - y < z - 2k \leq 2) \end{cases} \quad (3.17)$$

$$\vartheta(k) = 2 \sin \alpha\pi k \sin \alpha\pi(1-y) (\sin \alpha\pi)^{-1}$$

With solution (3.12) we can determine the form of the disturbed jet. If the equation of the lower limit is $Y = Y(x, t)$ and the solution is written in the form

$$Y = Y_1 \cos v\tau + Y_2 \sin v\tau$$

then from the condition (linearized) that the displacement velocities of the free surface and the flow velocity component normal to it are equal, we obtain the system of equations for Y_1, Y_2

$$M dY_1/dx + vY_2 = (\Phi_{1y})_{y=0}, \quad M dY_2/dx - vY_1 = (\Phi_{2y})_{y=0} \quad (3.18)$$

where Φ_{1y}, Φ_{2y} are functions which are known from (3.12) and (3.13). The boundary conditions are: $Y_1(0) = Y_2(0) = 0$. In the limiting case the solution for this system is easily found. The expressions for Y_1, Y_2 are found, generally speaking, to be rather cumbersome, so they will be given only in the case of integer and half-integer α . /91

For $\alpha = n$ ($n = 1, 2, \dots$)

$$\begin{aligned} Y_1^0 &= (\pi Y M^2 / P \beta^2) Y_1 = n^{-1} (2k+1) \sin n\pi z \\ Y_2^0 &= n^{-1} (2k+1) (\cos n\pi z - 1) \\ K_Y &= [(Y_1^0)^2 + (Y_2^0)^2]^{1/2} = n^{-1} (2k+1) |\sin^{1/2} n\pi z| \end{aligned} \quad (3.19)$$

For $\alpha = n - 1/2$

$$\begin{aligned} Y_1^0 &= (n - 1/2)^{-1} (-1)^k \sin (n - 1/2) \pi z \\ Y_2^0 &= (n - 1/2)^{-1} [(-1)^k \cos (n - 1/2) \pi z - (2k + 1)] \\ K_Y &= (n - 1/2)^{-1} [1 + (2k + 1)^2 + 2(2k + 1) (-1)^k \cos (n - 1/2) \pi z]^{1/2} \end{aligned} \quad (3.20)$$

4. Analysis of Results

A numerical study of solutions for (3.12) and (3.13) and a study of the limiting solution (3.16) shows that disturbances in velocities and pressure — i.e., quantities u_1 , u_2 , p_1 etc. — are piecewise continuous functions of coordinate x , y , which undergo a discontinuity at characteristics $y_1 = x/\beta - 2k$ and $y_2 = 2 - x/\beta - 2k$ (Figure 4). The discontinuities of these functions correspond to variations in the parameters of disturbed flow of compression and rarefaction waves which can be shown linearly by the indicated characteristics.

In the limiting case the dependence of the disturbance of gas dynamic parameters on coordinates x , y in the continuous portions is sinusoidal. Their dependence on x is periodic if α is a rational number. If α is stated in the form of an irreducible fraction $\alpha = a/b$ the period is equal to $2b$. For an irrational α , functions F_1 , F_2 (3.16) and their derivatives belong to the class of quasi-periodic functions. For $a/a_e < \infty$ the dependence of disturbances of velocities and pressure on x is nearly periodic, but the dependence on x and y in the continuous portions is nearly sinusoidal⁽¹⁾.

Let us proceed to the study of the coefficient of pressure disturbance amplification K_p . In the limiting case, as seen from (3.17), the dependence of K_p on the M number of the jet and frequency ν of the external acoustic waves is reduced to the dependence on a single parameter α . In the continuous portions K_p does not depend on coordinate z (or x). A detailed analysis shows

(1) The derivations are valid only for those frequencies ν (or α) for which solutions (3.12) or (3.16) have no singularities.

that the quantity K_p for a given value of k or for any value of the coordinate y is limited by the quantities

$$K_p \leq \begin{cases} 2k & (0 \leq z - 2k < y) \\ 2k + 1 & (y < z - 2k < 2 - y) \\ 2(k + 1) & (2 - y < z - 2k \leq 2) \end{cases} \quad (4.1)$$

For $k > 1$ the maximum possible value of K_p (4.1) is attained only for integer values of the parameter α (not for all y). Thus, the maximum amplification of disturbances in the jet is found when the external acoustic wave frequency coincides with one of the critical frequencies (3.14).

The dependence of the amplification coefficient K_p on the frequency α is shown in Figure 6a for the values $z = x/\beta = 4, 5, 6$ and $y = 0.25, 0.5, 0.75$ (curves 1, 2, 3 respectively).

At greater distances from the nozzle opening, the maximum value of K_p over α increases, and a sudden increase in K_p in the region of critical frequencies occurs in a narrower frequency interval $\Delta\alpha$, while for all other frequencies, pressure disturbances in the jet are of the same order of magnitude as at the boundary. For $\alpha = 1$, the curve $K_p(\alpha)$ for $y = 0.5$ reaches its maximum possible value. In the $\alpha = 1$ region, curves for $y = 0.25$ and $y = 0.75$ have a local maximum. The functions $K_p(\alpha)$ are periodic with a period of $T_\alpha = 2$ for $y = 0.5$, and $T_\alpha = 4$ for $y = 0.25$ and $y = 0.75$. The dependence of K_p on coordinate $z = x/\beta$ is given in Figure 6b for $y = 0.25, 0.5, 0.75$ (dotted, solid and dash-and-dotted lines) for $\alpha = 1$ and $\alpha = 0.5$ (for the latter α , only $z \geq 4$ is given to keep the figure from becoming too cumbersome). /92

Numerous calculations using Formulas (3.12) and (3.13) in the $2 \leq M \leq 4$, $0.5 \leq a/a_e \leq \infty$ range showed that the dependence of the coefficient of pressure disturbance amplification K_p on frequency ν of the jet Mach number M and the ratio of the speeds of sound a/a_e may be reduced (within an accuracy of 1 - 2%) to a dependence on a single parameter $\alpha = \nu/\nu_*$. Thus, all conclusions /93

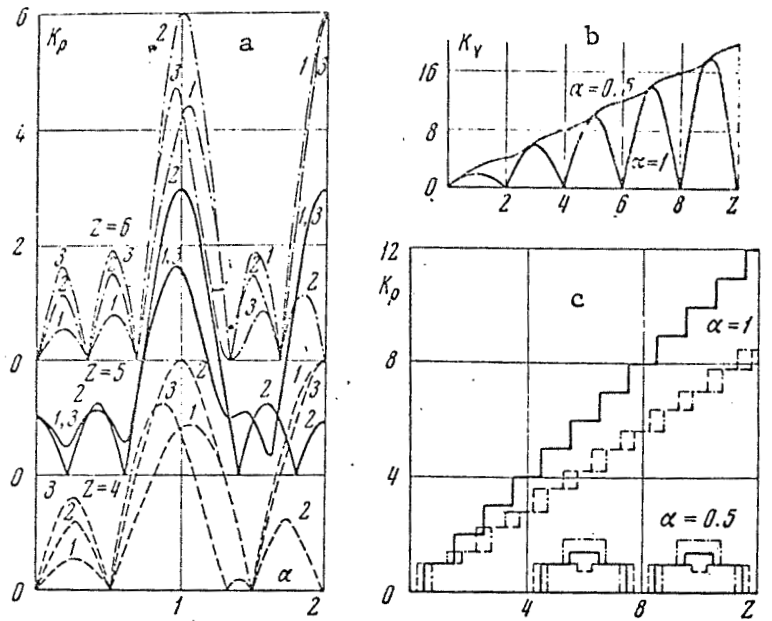


Figure 6.

regarding the behavior of K_p , drawn for the limiting case $a/a_e = \infty$, can be applied to the general case. However, it should be kept in mind that such a situation occurs only for amplitude values of pressure disturbances, whereas the quantities p , p_2 in the expression for pressure disturbance

$$p' = p_1 \cos \nu t + p_2 \sin \nu t$$

depend strongly on a/a_e .

In studying the expressions for the jet boundaries in the limiting case (3.19), (3.20), we note that at the critical frequency ($\alpha = 1$) the boundary points of the jet corresponding to integer even values of the coordinate z , remain stationary, and the maximum range of boundary oscillations occurs for integer, odd z . As the distance from the nozzle opening increases, these oscillations become more intense. Another type of oscillation occurs at $\alpha = 0.5$; in this case the quantity Y_1 is small in comparison with Y_2 , and the dependence of $Y_2(z)$ is almost linear on the average. Thus, the jet completes periodic oscillations around its undisturbed location as a whole. The dependence of the boundary oscillation amplitude of the jet K_Y on the coordinate z for $\alpha = 1$ and $\alpha = 0.5$ is shown in Figure 6c.

For the symmetric case ($j = 1$) calculations were made in the $2 \leq M \leq 4$, $0.5 \leq a/a_e \leq 2$ range. Numerical studies enabled us to state that here the dependence of coefficient K_p on the determining parameters is reduced to the

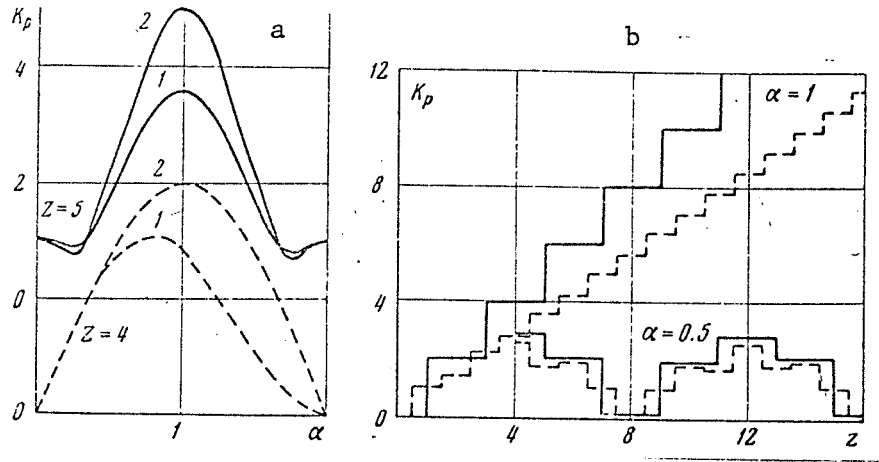


Figure 7.

dependence on the parameter $\alpha = v/v_*$, where v_* is found according to (3.15) for $n = 1$. The dependence $K_p(\alpha)$ for $z = 4$ and 5 and $y = 0.5$ and 1 is shown in Figure 7a (according to calculations for $M = 3$, $a/a_e = 1$), and Figure 7b shows the dependence $K_p(z)$ for the same y for $\alpha = 1$ and 0.5 . It can be seen that the maximum possible values of K_p coincide here with the corresponding values for the anti-symmetric case (4.1). However, if in the asymmetric case these values were reached at the points located on the jet periphery ($y = 0.5$ for $\alpha = 1$, $y = 0.25$ for $\alpha = 2$), then in the symmetric case this occurs only for the points of the middle jet line $y = 1$. As in the asymmetric case, the dependence $K_p(z)$ is a step dependence and for the critical frequency ($\alpha = 1$) K_p increases infinitely as the distance from the nozzle opening increases, while for $\alpha = 0.5$ it is nearly periodic.

Apart from a study of the linear solution, we also solved a nonlinear problem for the interaction of a jet with acoustic waves. Complete equations for gas dynamics were resolved numerically on the computer. The difference scheme used, as shown by methodical calculations and comparison with exact solutions, on the average, satisfactorily describe the flow of gas with discontinuity surfaces for small pressure drops in the latter (approximately 40%).

Figure 8 compares K_p for linear and nonlinear solutions in the symmetric case for $M = 3$, $a/a_e = 0.6$ for $z = 4$, $y = 1$. At the boundaries of the jet, the boundary conditions (3.2) with $j = 1$ were defined, and the quantity P was taken as equal to $P = 0.1p$, where p is the undisturbed pressure in the jet. At large values of time t , the solution was periodic with respect to t and did not depend on the initial data for $t = 0$. The program selected maximal and minimal values for pressure in accordance with time at the given points in the jet and computed their amplification coefficients

$$(K_p)_1 = P^{-1}(p_{\max} - p); \quad (K_p)_2 = P^{-1}(p - p_{\min}) \quad (4.2)$$

In the linear formulation $(K_p)_1 = (K_p)_2 = K_p$. For a nonlinear solution, /94 these quantities differ from one another (curves 1 and 2 in Figure 8). As in the linear solution, the dependence of K_p on frequency has peaks in the critical frequency region, while for the first critical frequency ($S = 0.11$) the linear (curve 0) and nonlinear solutions are rather close. At the second critical frequency ($S_* = 0.33$), for the nonlinear solution there is a conspicuous drop in the maximum of K_p compared to the first frequency.

The nonlinear solution enables us to study the dependence of the disturbance amplification coefficient on the amplitude P of the disturbing effect. The results of such a study are given in Figure 9 for the anti-symmetric case where $M = 4$, $a/a_e = 0.5$, $S = 1.15$, $S_* = 0.195$. We have shown the dependences of $(K_p)_1$ and $(K_p)_2$ (curves 1 and 2) on P for $z = 4$, $y = 0.4$ and 1 when $0.025 \leq P \leq 0.4$. When $y = 0.4$ (solid line), i.e., at the periphery of the jet, as P increases the value of K_p decreases, and may even become less than 1, i.e., disturbance in the jet is weakened rather than strengthened. These results corroborate the hypothesis made in Section 1 concerning the mechanism which limits the pulsations increase in the jet. When $y = 1$, i.e., on the median line of the jet (dotted line in Figure 9), the quantities K_p are small when P is small (in the linear solution they are equal to zero because of anti-symmetry). As P increases, the amplification coefficients increase and

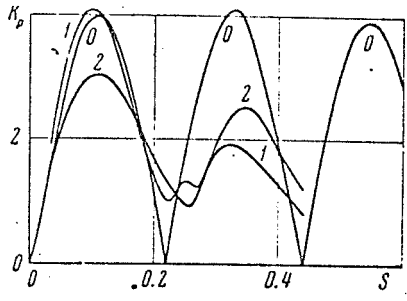


Figure 8.

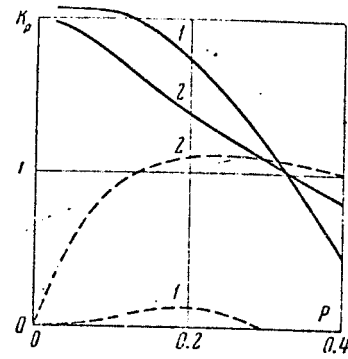


Figure 9.

consequently when the maximum is attained, they begin to decrease. The fact that the gas dynamic parameters of the median line of the jet are disturbed is explained by the fact that the variations in these parameters in compression and rarefaction waves are not equal.

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